THESIS FOR THE DEGREE OF MASTER OF SCIENCE

Drag Reduction Over Rough Surfaces

by

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Diploma Work

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Abstract

The main objective with this thesis is to derive drag force reduction, for moderate Reynolds numbers, when a microscopic rough surface of height ε is introduced over a flat plate. Such a problem is often impossible to treat with numerical methods since the microscopic scale is much smaller than the macroscopic scale which is present in the same equation. To overcome this problem, a homogenization process is performed on the rough surface and an effective (approximate) equation is obtained, which is an $\mathcal{O}(\varepsilon^2)$ approximation. This effective equation describes the flow at the macroscopic scale but it captures the flow characteristics in the microscopic scale through a constant coefficient operator. Given the effective equation, the drag force reduction is then derived.

Acknowledgement

I would like to express my deepest gratitude to my supervisor Professor Nils Svanstedt for his assistance and advice during my studies at Chalmers. His encouragement, support and valuable remarks were essential during this thesis work.

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Chapter 1

Introduction

It is well known that in a turbulent flow there exists a huge range of different scales of motion. These different scales of motion contributes to different characteristics of the flow. The largest, energy containing, scales are gigantic in comparison with the smallest, dissipative, scales. Interaction between these scales are crucial in the understanding how the flow behaves.

A typical example when these different scales interact is in the flow sourounding a golf ball. If we take a closer look at the golf ball, the surface is perforated and we see that the indentions in the center are slightly deeper than those at the poles. The corresponding wake behind the golf ball is dramatically changed if we compare it to a flat ball of the same size. We also know that the golf ball, with a proper backspin, would never fly as far as it does without the dimpled surface. Thus we can conclude that a rough surface can have a lower drag force than a smooth one.

During the last decade we have seen an increasing scientific interest in rough surfaces that reduces the drag. The most well known example of such a surface is the shark skin. Sharks belongs to one of the oldest living species and through millions of years of evolution, they have developed a drag reducing skin. The skin consists of dermal denticles (see figure 1.1) where the smallest structures are in the range of 0.01-0.1mm.

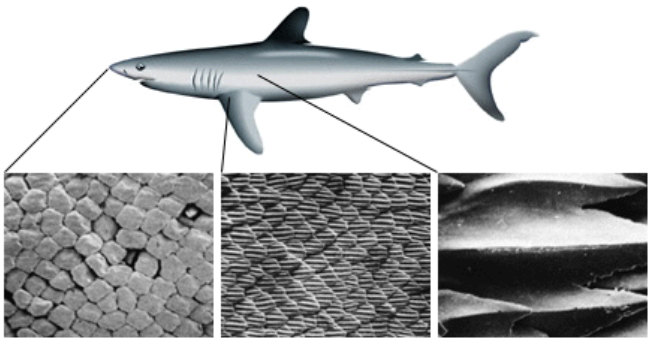


Figure 1.1: The shark skin consists of small dermal denticles.

In this thesis we try to explain why the drag reducing effect appears for a rough surface. We do this by introducing a rough surface on a flat plate and then compare the drag with corresponding flat plate. To be able to prove this, we make some restriction. First, we only consider flows with moderate Reynolds numbers. Second, we do not consider flow separation. Lastly, we consider only skin friction, not form drag.

The outline of this thesis is as follows. We start, in chapter 2, with an introduction to homogenization theory. In chapter 3, we explore the theoretical frame work that is needed throughout the thesis. Next, we describe the problem and the rough surface in detail in chapter 4-5 and in chapter 6-9, we perform a homogenization process on the rough surface. These chapters also include existence, uniqueness and convergence results obtained in the homogenization process. In chapter 10, we explain the basic theory of boundary-layers. Further, in chapter 11, we give the boundary-layer equation which is the equation describing the flow at the tip of the rough surface. Next, in chapter 12, we present the effective equation where the flow at the rough surface is replaced with an artificial smooth boundary along with the constant coefficient operator. This operator captures the flow characteristics at the rough surface and is evaluated at the artificial smooth boundary. In chapter 13, we present numerical simulations of the constant coefficient operator C_*^a which is computed in Comsol Multiphysics. Finally, in chapter 14, we summarize the main conclusions of this thesis.

Chapter 2

Homogenization Theory

The theory of homogenization deals with the study of processes which take place in heterogenous periodic structures, such as flows over a (periodic) rough surface. In such flows there exists (at least) two length scales: a microscopic scale associated with the characteristic height of the rough surface and a macroscopic scale associated with the mean flow. When the rough surface is very small compared to the macroscopic mean flow (such as in the case when riblets are attached to air planes) it is natural to introduce a small parameter ε which is the ratio between the scales.

Quite often, the mathematical model describing the phenomena in the microscopic structures is given by some physics law involving the parameter ε. For small values of ε the numerical treatment of the macroscopic model is very difficult, often impossible. On the other hand, in applications, one usually is not interested in what happens exactly on the microscopic scale; one rather needs macroscopic (effective) approximations which take into account the local effects.

The process of homogenization consist in deriving such macroscopic descriptions by performing an asymptotic analysis with the scale parameter ε tending to zero. More precisely, we start with family of operators $\mathcal{A}_\varepsilon$, depending on ε, whose coefficients are periodic functions on a domain Ω. The law which includes the microscopic variation is given by the ε-problem:

$$\mathcal{A}_\varepsilon u_\varepsilon = f \quad \text{in } \Omega$$

with u_ε subjected to appropriate boundary conditions. The method of homogenization leads to a macroscopic law

$$\mathcal{A}_0 u_0 = f \quad \text{in } \Omega$$

with a constant coefficient operator $\mathcal{A}_0$ (in this thesis our constant coefficient operator will turn out to be the Navier constant C_*). This macroscopic law, together with appropriate boundary conditions for u_0, determines a unique macroscopic approximation for u_ε, with the property that the sequence u_ε tends to u_0 in properly chosen function spaces as $\varepsilon \to 0$. Next, one has to study the accuracy of the approximation u_0 by giving estimates on the rate of convergence.

Chapter 3

Theoretical Framework

In order to study homogenization processes, we need to consult the tools developed in functional analysis and theory developed for partial differential equations (PDE). This section is just a brief review of the most central definitions and theorems (without proofs) that are used throughout this thesis. It is also meant to introduce the reader to our notation.

3.1 Banach and Hilbert spaces

Definition 3.1.1 (Banach space). *A Banach space X is a complete normed vector space, i.e., for all $u, v \in X$ and for $\lambda \in \mathbb{R}$, it satisfies:*

(i) $\|u\|_X \geq 0$, *if* $\|u\|_X = 0 \Leftrightarrow u \equiv 0$

(ii) $\|\lambda u\|_X = |\lambda| \|u\|_X$

(iii) $\|u + v\|_X \leq \|u\|_X + \|v\|_X$

(iv) every Cauchy sequence in X converges strongly to an element in X.

Definition 3.1.2. *Let X be a Banach space over $\mathbb{R}$ and let $|\cdot|$ denote the absolute value, which defines a norm on $\mathbb{R}$. The mapping $\mathcal{A} : X \to \mathbb{R}$ is a bounded linear operator, denoted $\mathcal{A} \in \mathcal{B}(X, \mathbb{R})$, provided that*

(i) $\|\mathcal{A}u\|_X \leq M|u|, \quad \forall\, u \in X, \quad for\ some\ constant\ 0 < M < \infty$

(ii) $\mathcal{A}(\lambda u + \gamma v) = \lambda \mathcal{A}u + \gamma \mathcal{A}v, \quad \forall\, u, v \in X,\ \lambda, \gamma \in \mathbb{R}$

Definition 3.1.3. *The dual space of a Banach space X is denoted X^* and is defined as $X^* = \mathcal{B}(X, \mathbb{R})$. Moreover, X is called reflexive if $(X^*)^* = X$.*

Definition 3.1.4 (Hilbert space). *A Hilbert space X is a Banach space with an inner product, i.e., a Banach space such that for all $u, v, f \in X$ and $\lambda, \gamma \in \mathbb{C}$ or $\mathbb{R}$, it satisfies:*

(i) $(u,u)_X \geq 0$, *if* $(u,u)_X = 0 \Leftrightarrow u \equiv 0$

(ii) $(\lambda u + \gamma v, f)_X = \lambda(u,f)_X + \gamma(v,f)_X$

(iii) $(u,v)_X = \overline{(v,u)}_X$.

We notice that for a complex Hilbert space the inner product is anti-symmetric with respect to the second argument. A norm on a Hilbert space X is defined as

$$(u,u)_X = \|u\|_X^2.$$

When we consider an inner product between dual pairs ,i.e., for $u \in X^*$ and $v \in X$, we write the inner product $(u,v)_{X^*,X}$ as $< u, v >$. Furthermore, all elements u, v in a Hilbert space X satisfies the Cauchy-Schwarz inequality:

$$|(u,v)_X| \leq \|u\|_X \|v\|_X.$$

3.2 Function spaces

Let Ω be an open set in $\mathbb{R}^N$. We will denote by $C(\Omega)$ the space of continuous functions $u : \Omega \to \mathbb{R}$. This space, when equipped with the supremum norm

$$\|u\|_{C(\Omega)} = \sup_{x \in \Omega} |u(x)|,$$

is a Banach space. Similarly, we can define the space $C^k(\Omega)$ of the k-times continuous differentiable functions. The notation $C^\infty(\Omega)$ will be used to denote the space of smooth functions. We will denote the space of smooth functions with compact support in Ω by $C_0^\infty(\Omega)$.

3.2.1 L^P spaces

Let $1 \leq p < \infty$ and let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the L^p-norm by

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Remark 3.2.1. *For $p = +\infty$ we define the norm as*

$$\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x \in \Omega} |u|.$$

Theorem 3.2.2. *Some basic properties of L^p spaces:*
(i) The vector space $L^p(\Omega)$, equipped with the L^p-norm, is a Banach space for every $p \in [1, +\infty]$.

(ii) $L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx, \qquad \forall\, u, v \in L^2(\Omega).$$

(iii) $L^p(\Omega)$ is reflexive for $p \in (1, +\infty)$.

Remark 3.2.3. *The physical interpretation of the space $L^2(\Omega)$ is that if we regard u as a velocity, then for $u \in L^2(\Omega)$ the kinetic energy in the domain Ω is finite.*

3.2.2 Sobolev spaces

In order to define Sobolev spaces we need to define the weak derivative.

Definition 3.2.4. *Let $u, v \in L^2(\Omega)$. We say that v is the first weak derivative of u with respect to x_i if*

$$\int_\Omega u\frac{\partial \varphi}{\partial x_i}dx = -\int_\Omega v\varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Throughout this thesis, φ will denote test functions of appropriate type. We also use the notation $\nabla u = \frac{\partial u}{\partial x_i} n_i$ for n_i being the normal vector in the i:th direction. Also, $\frac{\partial u}{\partial n}$ will denote $\nabla u \cdot n_i$. Finally, we denote the Laplacian as $\Delta u = \text{div } \nabla u$.

Definition 3.2.5. *The Sobolev space $H^1(\Omega)$ consists of all square integrable functions from Ω to $\mathbb{R}$ whose first order weak derivatives exist and are square integrable:*

$$H^1(\Omega) = \{u \in L^2(\Omega) :\ \nabla u \in L^2(\Omega)\}.$$

The space $H^1(\Omega)$ is a Hilbert space with norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

and an inner product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}.$$

Definition 3.2.6. *The Sobolev space $H_0^1(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the H^1-norm.*

A very important property of the $H_0^1(\Omega)$ is the fact that we can control elements in L^2 in terms of the L^2-norm of their gradients via the Poincaré inequality.

Theorem 3.2.7 (Poincaré inequality). *Let Ω be open bounded set in $\mathbb{R}^N$. Then there exist a constant C, which only depends on Ω, such that*

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \qquad for\ all\ u \in H_0^1(\Omega).$$

An immediate result of the Poincaré inequality is that

$$\|u\|_{H_0^1(\Omega)} \sim \|\nabla u\|_{L^2(\Omega)},$$

i.e., equivalent norms. This property will be extremely important when we use duality arguments to prove uniqueness and existence of solutions. We can then identify $L^2(\Omega)$ as the dual space of $H_0^1(\Omega)$.

3.2.3 Convergence

In this section we give the definitions of the three types of convergence that are being used in this thesis. The so called *two-scale convergence* is an essential tool when studying homogenization theory. This concept was introduced by Gabriel Nguetseng in 1989. For further reading we strongly recommend [12].

Definition 3.2.8. *A sequence $\{u_n\}_{n=1}^{\infty} \subset X$ is strongly convergent, denoted $u_n \to u$ in X, provided that*

$$\lim_{n\to\infty} \|u_n - u\|_X = 0.$$

Definition 3.2.9. *A sequence $\{u_n\}_{n=1}^{\infty} \subset X$ is weakly convergent, denoted $u_n \rightharpoonup u$ in X, provided that*

$$\lim_{n\to\infty} (u_n, u)_X = \|u\|_X^2.$$

Corollary 3.2.10. *Let X be a reflexive Banach space. Let $\{u_n\} \subset X^*$ be a sequence weakly convergent towards $u \in X^*$ and let*

$$< u_n, v_n > \to < u, v >, \quad \forall \{v_n\} \subset X,$$

converging weakly in X towards $v \in X$. Then $u_n \to u$ in X^* strongly.

In order to define two-scale convergence we need to consider a particular type of test functions, an *admissible test function*.

Definition 3.2.11. *We say that the test function $\varphi \in L^2(\Omega \times \mathbf{Y})$ is admissible if*

$$\lim_{\varepsilon\to 0} \int_\Omega \left|\varphi\left(x, \frac{x}{\varepsilon}\right)\right|^2 dx = \int_\Omega \int_{\mathbf{Y}} |\varphi(x,y)|^2 dy dx.$$

We notice that an admissible test function asymptotically decouples the dependence of x and $\frac{x}{\varepsilon}$ in the L^2-norm. We are now ready to define the two-scale convergence.

Definition 3.2.12. *Let u_ε be a sequence in $L^2(\Omega)$. We will say that u_ε two-scale converges to $u_0(x,y) \in L^2(\Omega \times \mathbf{Y})$, denoted $u_\varepsilon \overset{2}{\rightharpoonup} u_0$, if for every admissible test function $\varphi \in (\Omega \times \mathbf{Y})$ we have*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_{\mathbf{Y}} u_0(x,y) \varphi(x,y)\, dy dx.$$

We continue with the following powerful theorem which states that if we have a bounded sequence in L^2 we can always extract a subsequence which two-scale converges in the L^2-norm.

Theorem 3.2.13. *From each bounded sequence $u_\varepsilon \in L^2(\Omega)$ one can extract a subsequence, still denoted u_ε, with two-scale convergence in $L^2(\Omega)$ to a $u_0 \in L^2(\Omega \times \mathbf{Y})$.*

An immediate important consequence of the two-scale convergence is the result of averaging. We have the following lemma.

Lemma 3.2.14 (Averaging). *Let* $u_\varepsilon \overset{2}{\rightharpoonup} u_0$. *Then*

$$u_\varepsilon \rightharpoonup \overline{u}_0(x) := \int_{\mathbf{Y}} u_0(x,y)\, dy.$$

This lemma tells us that the fluctuations on the microscopic scale y converge to an average value in limit process as $\varepsilon \to 0$. As we will see later, this is the link between the macroscopic scale x and the microscopic scale y and thus a fundamental result in homogenization theory. But a obvious question arises: which $\bar{u}_0(x)$ does this sequence u_ε converge to? As will see in a moment, if we pick a sequence u_ε to be a *multi scale expansion* we gain knowledge about $\bar{u}_0(x)$.

Definition 3.2.15 (Multi scale expansion). *We define* $u_\varepsilon \in L^2(\Omega)$ *as a multi scale expansion for*

$$u_\varepsilon = u_0\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \tfrac{x}{\varepsilon}\right) + \dots \quad ,$$

where $u_j(x,y) \in L^2(\Omega \times \mathbf{Y}),\ j = 0, 1, \dots$.

If we choose the multi scale expansion to be periodic in the microscopic variable y we can use the results of the two-scale convergence. We have then the following lemma.

Lemma 3.2.16. *Consider* $u_\varepsilon \in L^2(\Omega)$ *being a multi scale expansion where* $u_j(x,y) \in L^2(\Omega; H^1_{per}(\mathbf{Y}))$ *for* $j = 0, 1, \dots$, Ω *being a bounded domain in* $\mathbb{R}^N$. *Then*

$$u_\varepsilon \overset{2}{\rightharpoonup} u_0.$$

Thus, if we have a multi scaled sequence $u_\varepsilon(x) \in L^2(\Omega)$ such that $u_j(x,y) \in L^2(\Omega; H^1_{per}(\mathbf{Y}))$ for $j = 0, 1, \dots$, we conclude that, together with the result of averaging,

$$u_\varepsilon \overset{2}{\rightharpoonup} \bar{u}_0(x).$$

3.2.4 Equations in Hilbert spaces

Throughout this thesis we will frequently encounter linear PDEs in a Hilbert space setting. It is consequently useful to develop an abstract formulation of such problems. We summarize this theory here. There are two main result that are of great importance: the Lax-Milgram existence theory and the Fredholm alternative.

3.2.5 Lax-Milgram theory

Let X be a Hilbert space with an inner product $(\cdot,\cdot)_X$ and let $\mathcal{A} : X \to X^*$ be a linear operator. Let $f \in X^*$ and let $< \cdot, \cdot >$ be the dual paring between X^* and X. We are interesting in studying the equation

$$\mathcal{A}u = f. \tag{3.1}$$

An equivalent formulation of (3.1) is

$$(\mathcal{A}u, \varphi)_X =< f, \varphi >, \qquad \forall \varphi \in X. \tag{3.2}$$

The linearity of $\mathcal{A}$ implies that the left hand side of the above equation defines a *bilinear* functional $a : X \times X \to \mathbb{R}$ such that

$$a[u, \varphi] =< f, \varphi >, \qquad \forall \varphi \in X. \tag{3.3}$$

Existence and uniqueness of solutions of the form (3.3) can be proved by the means of the Lax-Milgram theorem, which is an extension of the Riesz representation theorem. These theorems implies that every Hilbert space is reflexive. Thus, we can identify the dual space with itself through the Lax-Milgram theorem.

Theorem 3.2.17 (Lax-Milgram). *Let X be a Hilbert space with norm $\| \cdot \|_X$ and let $< \cdot, \cdot >$ denote the dual pairing between X^* and X. Moreover, let $a : X \times X \to \mathbb{R}$ be a bilinear functional with the following properties:*
(i) (Bounded) There exist a constant $C > 0$ such that

$$a[u, \varphi] \leq C\|u\|_X \|\varphi\|_X, \qquad \forall u, \varphi \in X.$$

(ii) (Coercive) There exist a constant $K > 0$ such that

$$a[u, u] \geq K\|u\|_X^2, \qquad \forall u \in X.$$

Then, for all $f \in \mathcal{B}(X, \mathbb{R}) = X^*$, there exists a unique element $u \in X$ such that

$$a[u, \varphi] =< f, \varphi > \qquad for\ all\ \varphi \in X.$$

3.2.6 Fredholm alternative

In order to examine the Fredholm alternative, we need to define some properties of operators.

Definition 3.2.18. *We identify the adjoint operator of $\mathcal{A}$ as $\mathcal{A}^\dagger$ through*

$$(\mathcal{A}u, \varphi) = (u, \mathcal{A}^\dagger \varphi).$$

Definition 3.2.19. *$\mathcal{A}$ is called self-adjoint if $\mathcal{A} = \mathcal{A}^\dagger$.*

Definition 3.2.20 (Compact operator). *Let $\{u_n\}_{n=1}^\infty \subset X$ be a Cauchy sequence and let $\mathcal{A}$ be a continuous operator, i.e.*

$$\lim_{n \to \infty} \|\mathcal{A}u_n - \mathcal{A}u\|_X = 0.$$

If there exist a sub Cauchy sequence $\{u_{n_k}\}_{k=1}^\infty \subset X$ such that

$$\lim_{k \to \infty} \|\mathcal{A}u_{n_k} - \mathcal{A}u_n\|_X = 0$$

then $\mathcal{A}$ is a compact operator.

Assume that X is a Hilbert space and that $\mathcal{A} : X \to X$ is a linear operator. Consider the equation (3.1) with $f \in X$. The adjoint operator $A^\dagger$ is then defined as

$$(\mathcal{A}u, \varphi)_X = (u, \mathcal{A}^\dagger\varphi)_X, \qquad \forall\, u, \varphi \in X.$$

Let now $\varphi \in X$ belong to the null space of $\mathcal{A}^\dagger$, i.e.,

$$\mathbf{N}(\mathcal{A}^\dagger) = \{\varphi \in X :\ \mathcal{A}^\dagger\varphi = 0\}.$$

If we consider this choice of test function φ, equation (3.2) simplifies to

$$(f, \varphi)_X = 0, \qquad \forall\, \varphi \in \mathbf{N}(\mathcal{A}^\dagger).$$

Consequently, a necessary condition for the existence of a solution to equation (3.1) is that the right hand side is orthogonal to the null space of the adjoint operator of $\mathcal{A}$. The above formal argument can be extended to the case when $\mathcal{A}$ is a compact perturbation of the identity: $\mathcal{A} = \mathcal{I} - \mathcal{K}$, with $\mathcal{K}$ compact. This theory is summarized in the following theorem.

Theorem 3.2.21 (**Fredholm alternative**). *Let X be a Hilbert space and let $\mathcal{K} : X \to X$ be a compact operator. Then the following holds:*
(i) Either the equations

$$(\mathcal{I} - \mathcal{K})u = f \tag{3.4}$$

$$(\mathcal{I} - \mathcal{K}^\dagger)U = F \tag{3.5}$$

have unique solutions for every $f, F \in X$ ***or***

(ii) the homogeneous equations

$$(\mathcal{I} - \mathcal{K})u_0 = 0 \tag{3.6}$$

$$(\mathcal{I} - \mathcal{K}^\dagger)U_0 = 0 \tag{3.7}$$

have the same finite number of non-trivial solutions:

$$dim(\mathbf{N}(\mathcal{I} - \mathcal{K})) = dim(\mathbf{N}(\mathcal{I} - \mathcal{K}^\dagger)) < \infty.$$

In this case equations (3.4) and (3.5) have a solution if and only if

$$(f, U_0)_X = 0 \qquad \forall\, U_0 \in \mathbf{N}(\mathcal{I} - \mathcal{K}^\dagger)$$

and

$$(F, u_0)_X = 0 \qquad \forall\, u_0 \in \mathbf{N}(\mathcal{I} - \mathcal{K}).$$

Chapter 4

Problem Description

In this thesis we try to give an answer to the following problem:

Can we obtain a reduction of the tangential drag force $\mathcal{F}_t$ if we introduce a periodic rough surface with a characteristic height of ε on a flat plate?

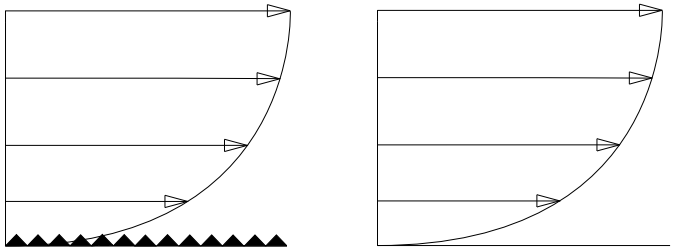


Figure 4.1: The left picture illustrates a flow over a plate with a rough surface whilst the right picture illustrates the same flow over a flat plate.

If we associate u_ε as the solution for the flow over the rough surface and u with flow over the flat plate we then seek the difference

$$|\mathcal{F}_t(u_\varepsilon) - \mathcal{F}_t(u)|. \tag{4.1}$$

As mentioned above, the solution u_ε is not feasible (for small ε) when regarding numerical simulations. If we want to investigate (4.1) numerically, we need to find an approximate solution to u_ε. Through a homogenization process we can find the so called *effective* equation that describes u_ε in the macroscopic scale (x) where the flow characteristics of the microscopic scale (y) is captured in a constant coefficient operator when $\varepsilon \to 0$.

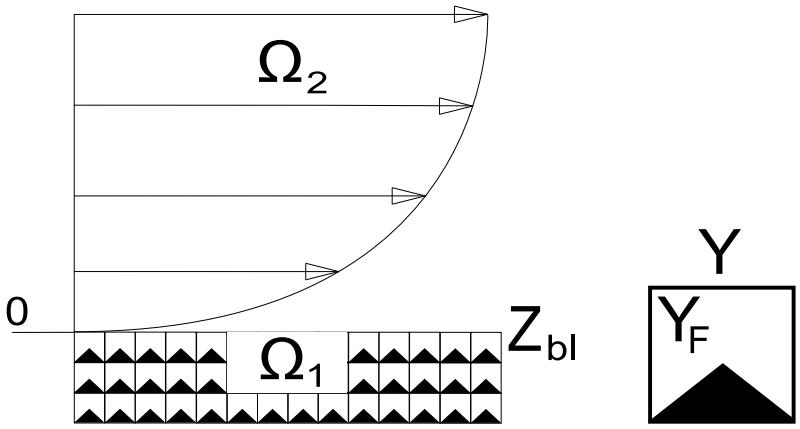


Figure 4.2: Illustration of the different regions. The rough surface Ω_1 is built up by unit cells $\mathbf{Y}$ which consists of two parts: $\mathbf{Y_F}$ which represents the fluid part and $\mathbf{S}$ which represents the the solid (rough) part.

To derive the effective equation, $u^e(x)$, we construct a computational domain that consists of separate regions where we can develop equations that fit the local problems. Once these equations are established, we then start the work to combine these equations to finally get the effective equation.

The different regions that we divide our flow problem in are (see figure 4.2):

- Ω_1 - the rough surface
- Ω_2 - the mean flow
- Z_{bl} - the interface between Ω_1 and Ω_2

The rough surface is constructed by unit cells $\mathbf{Y}$. In order to build a microscopic rough surface of height ε, we need to scale the unit cells. This is explained in detail in chapter 5. The interface between Ω_1 and Ω_2 is illustrated in figure 4.3.

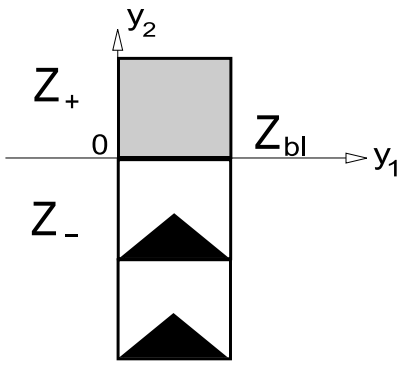


Figure 4.3: The interface between Ω_1 and Ω_2. We denote $Z_* = Z_- \cup Z_{bl} \cup Z_+$. Z_- is the unit cell closest to $y_2 = 0$ in Ω_2. Z_+ (the grey unit cell) is then a unit cell in Ω_2.

Having these pictures in mind, we want to derive the effective equation $u^e(x)$ which is valid in $\Omega_2 \cup Z_-$. This is to be done in the following way:

- Find the equation for v_ε through a homogenization process which describes the flow in the microscopic scale in $\Omega_1 \setminus Z_-$
- Find the equation for u_ε which describes the flow in $\Omega_2 \cup Z_-$ in both the microscopic and macroscopic scale.
- Find the equation for $\omega_*(y)$ which describes the flow in the microscopic scale in Z_*.
- Finally, combine the above result to replace u_ε with the effective $u^e(x)$ which describes the flow only at the macroscopic scale x in $\Omega_2 \cup Z_-$ but captures the flow characteristics at the microscopic scale y in Z_- through a constant coefficient operator. $u^e(x)$ is then solved exclusively in Ω_2 where the rough surface is replaced by an artificial smooth surface Z_{bl} and the no-slip boundary condition on the rough surface is replaced with the constant coefficient operator $-\varepsilon C_* \frac{\partial u_i^e(x)}{\partial x_2}$, i.e., the Navier slip condition, on Z_{bl}.

We proceed in the next chapter by constructing the rough surface Ω_1.

Chapter 5

The Rough Surface Ω_1

Let Ω_1 be an open bounded set in $(0, L) \times (0, L)$ with a boundary $\partial\Omega_1$. We want to define the geometrical structures that defines the rough surface inside the unit cell. This structure will be a periodic arrangement.

Let $\mathbf{Y} = (0, 1) \times (0, 1)$ be the unit cell in Ω_1. We denote the solid part , i.e., the rough surface, $\mathbf{S}$ which is a closed subspace of $\mathbf{Y}$. The remainder of the unit cell is then the fluid which we denote $\mathbf{Y_F}$. Obviously, $\mathbf{Y_F} = \mathbf{Y} \setminus \mathbf{S}$. Thus is the open set Ω_1 divided into a number of unit cells containing a rough surface and a fluid, hence $\mathbf{Y} \subset \Omega_1$ where $\mathbf{Y} = \mathbf{S} \cup \mathbf{Y_F}$. As we mentioned earlier we want to make the rough surface $\mathbf{S}$ periodic in Ω_1. In order to do this we introduce the periodic repetition of $\mathbf{S}$ all over $\mathbb{R}^2$ as

$$\mathbf{S}^k = \mathbf{S} + k, \qquad k \in \mathbb{Z}^2.$$

Thus we obtain a closed set of all rough surfaces in Ω_1 , denoted $\mathbf{E}_S$, as

$$\mathbf{E}_S = \bigcup_{k \in \mathbb{Z}^2} \mathbf{S}^k.$$

In the same way we define all fluid contained in Ω_1 as

$$\mathbf{E}_F = \mathbb{R}^2 \setminus \mathbf{E}_S.$$

We have thus created a mesh in Ω_1 of unit cells that consist of a rough surface that is periodic all over $\mathbb{R}^2$. Next step is to make this cells and the rough surfaces within the small cells. To be more precise, we want the structures to be dependent on a parameter ε. For simplicity we assume that $\frac{\varepsilon}{L} = \mathbb{Z}$. Thus we shrink the surface and the fluid contained in the unit cells. Before we can do that, we need to make some additional assumptions regarding $\mathbf{E}_F$ and $\mathbf{E}_S$.

- $\mathbf{E}_F$ is an open connected set of strictly positive measure with a Lipschitz boundary and $\mathbf{E}_S$ has a positive measure in $\bar{\mathbf{Y}}$.
- $\mathbf{E}_F$ and the interior of $\mathbf{E}_S$ are open sets with boundary of class C_0^1,which are locally located on one side of the boundary. Moreover, $\mathbf{E}_F$ is connected.

With the above assumptions we can define a linear homeomorphism $\prod_\varepsilon^i$, i.e., a similar transformation which preserves orientation. With this transformation we can shrink the

rough surface and the fluid part contained in $\mathbf{Y}$ into a cell of size $\varepsilon \times \varepsilon$. Thus for a given $\varepsilon > 0$, Ω_1 is then divided into a regular mesh size of length ε and each cell we denote $\mathbf{Y}_\varepsilon^i$ with $1 \leq i \leq N(\varepsilon) = |\Omega_1|\varepsilon^{-2}[1 + \mathcal{O}(1)]$. Thus each cell $\mathbf{Y}_\varepsilon^i$ is homeomorphic to $\mathbf{Y}$ with a homothety ratio of ε^{-2}. Now we can describe the rough surface within $\mathbf{Y}_\varepsilon^i$ as

$$\mathbf{S}_\varepsilon^i = \left(\prod_\varepsilon^i\right)^{-1} (\mathbf{S})$$

and the fluid part within $\mathbf{Y}_\varepsilon^i$ as

$$\mathbf{Y}_{\mathbf{F}_\varepsilon^i} = \left(\prod_\varepsilon^i\right)^{-1} (\mathbf{Y}_\mathbf{F}).$$

It is convenient to define a closed subset of Ω_1 as a cluster of rough surfaces. We denote this cluster as $\mathbf{T}_\varepsilon$ and define this as

$$\mathbf{T}_\varepsilon = \{k \in \mathbb{Z}^N \ : \mathbf{S}_\varepsilon^k \subset \Omega_1\}.$$

In a similar manner, we define

$$\mathbf{O}_\varepsilon = \bigcup_{k \in \mathbf{T}_\varepsilon} \mathbf{S}_\varepsilon^k$$

which is then the total part of the rough surface contained in Ω_1. Finally, we define all fluid part contained in Ω_1 as

$$\Omega_\varepsilon = \Omega_1 \setminus \mathbf{O}_\varepsilon = \Omega_1 \cap \varepsilon \mathbf{E}_F.$$

Chapter 6

Stokes' Equation in Ω_1

With the geometry defined for the rough surface, we need an appropriate equation to describe the flow in Ω_ε, which is the fluid contained in Ω_1. We consider the steady, newtonian, incompressible Navier-Stokes equations, i.e.,

$$(v \cdot \nabla)v - \nu\Delta v + \nabla q = f \quad \text{in } \Omega_\varepsilon \tag{6.1}$$

$$\text{div } v = 0 \quad \text{in } \Omega_\varepsilon \tag{6.2}$$

where v is the velocity, q the pressure, ν the kinematic viscosity and f a source term. For convenience, we have a unit density. In order to make use of the homogenization process in Ω_1, we need to modify the given Navier-Stokes equations.

The momentum equation (6.1) in Ω_ε will differ whereas the continuity equation (6.2) will be the same in Ω_ε. The flow is regarded as incompressible and thus is (6.2) valid in Ω_ε. But for the momentum equation we need some physical argument to find the right equation.

As mentioned in the introduction, the flow is highly dependent on the scales of motion. The flow in Ω_2 is associated with the macroscopic scale x whilst the flow in Ω_ε is associated with the microscopic scale y. If we apply a Fourier transform on (6.1) we observe that

$$\mathrm{F}_\kappa[-\nu\Delta u] = \kappa^2\nu\ \hat{u}, \tag{6.3}$$

where κ is the wave number. Since $\kappa \propto \frac{1}{l}$ for some length scale l, we conclude that this term is the dominating term in Ω_ε where the flow is characterized by the small scale $y \sim l \ll 1$. Note that we have not given an argument if (6.3) exist, it is just a argument from a physical point of view. This conclusion is in the agreement of the so called *energy cascade* process [1]. In the energy cascade the turbulent kinetic energy are transferred from the large scale into smaller and smaller scales and is then finally dissipated.

From the above argument, we conclude that the non-linear term in (6.1) is neglected in Ω_ε since the Laplacian term will be dominating in the microscopic scales. The pressure term remains since this, together with f, is the driving force of the velocity field. Thus we have, together with the no-slip condition on the rough surface, the *Stokes' equations*

$$\begin{cases} \mathcal{A} := -\nu\Delta v + \nabla q = f & \text{in } \Omega_\varepsilon \\ \text{div } v = 0 & \text{in } \Omega_\varepsilon \\ v = 0 & \text{on } \partial\Omega_\varepsilon \setminus Z_- \\ \{v, q\} \quad \text{is } L-\text{periodic.} & \end{cases} \tag{6.4}$$

Chapter 7

Existence and Uniqueness of the Stokes' Equation

Having defined the equation governing the rough surface, we need to investigate its properties. Equation (6.4) is, as we concluded, valid in Ω_ε and in this section we prove the uniqueness and existence of (6.4).

7.1 Existence and Uniqueness of the Velocity v

Proof. This proof is based on the Lax-Milgram theorem and this type of argument will be used many times during this thesis. However, we will only show all steps in this case as an illustrative example.

Clearly, according to the Lax-Milgram theorem, we need to find a suitable Hilbert space that both fulfill equation (6.4) and the requirements in the Lax-Milgram theorem. If we multiply (6.4) with a test function φ and integrate over Ω_ε we get

$$-\nu \int_{\Omega_\varepsilon} \Delta v \, \varphi \, dx + \int_{\Omega_\varepsilon} \nabla q \, \varphi \, dx = \int_{\Omega_\varepsilon} f \, \varphi \, dx. \tag{7.1}$$

Using Green's theorem on the left hand side we obtain

$$\nu \int_{\Omega_\varepsilon} \nabla v \, \nabla \varphi \, dx - \nu \int_{\partial\Omega_\varepsilon} \frac{\partial v}{\partial n} \, \varphi \, ds - \int_{\Omega_\varepsilon} q \, \text{div} \, \varphi \, dx + \int_{\partial\Omega_\varepsilon} q \cdot n \, \varphi \, ds = \int_{\Omega_\varepsilon} f \, \varphi \, dx.$$

If we choose $\varphi \in V(\Omega_\varepsilon)$ where $V(\Omega_\varepsilon) = \{\varphi \in H_0^1(\Omega_\varepsilon) : \text{div } \varphi = 0 \text{ in } \Omega_\varepsilon\}$, (7.1) is simplified to

$$\nu \int_{\Omega_\varepsilon} \nabla v \, \nabla \varphi \, dx = \int_{\Omega_\varepsilon} f \, \varphi \, dx, \qquad \forall \, \varphi \in V(\Omega_\varepsilon). \tag{7.2}$$

Clearly, the boundary conditions for v is satisfied for $v \in V(\Omega_\varepsilon)$. Thus we can identify the left hand side in (7.2) as an inner product on $H_0^1(\Omega_\varepsilon)$. Since $f \in L^2(\Omega_\varepsilon)$, we identify the right hand side in (7.2) as a dual inner product between f and φ. We rewrite (7.2) as

$$a[v, \varphi] = < f, \varphi > . \tag{7.3}$$

To prove that there exist a unique $v \in V(\Omega_\varepsilon)$ we need to prove that the functional $a : V(\Omega_\varepsilon) \times V(\Omega_\varepsilon) \to \mathbb{R}$ is bilinear, bounded and coercive and that $f \in \mathcal{B}(L^2(\Omega_\varepsilon), \mathbb{R}) = V(\Omega_\varepsilon)^*$. Clearly,

$$\lambda a[v, \varphi] = a[\lambda v, \varphi] = a[v, \lambda\varphi], \qquad \forall\, \lambda \in \mathbb{R},$$

hence bilinear. Moreover,

$$a[v, \varphi] = \nu \int_{\Omega_\varepsilon} \nabla v \, \nabla\varphi \, dx = \nu \, (\nabla v, \nabla\varphi)_{L^2(\Omega_\varepsilon)} \leq |\nu \, (\nabla v, \nabla\varphi)_{L^2(\Omega_\varepsilon)}| \leq$$

$$\leq \nu \, \|\nabla v\|_{L^2(\Omega_\varepsilon)} \|\nabla\varphi\|_{L^2(\Omega_\varepsilon)} \leq C \|v\|_{V(\Omega_\varepsilon)} \|\varphi\|_{V(\Omega_\varepsilon)},$$

thus bounded. Note that throughout this thesis, we will refer C as an arbitrary constant which will vary from time to time. To be able to prove the coericivity of $a[\cdot, \cdot]$ we need to consult Fredrich's first inequality.

Corollary 7.1.1 (Fredrich's first inequality). *Let $w \in H_0^1(\Omega)$. There exists a positive constant α such that*

$$\int_\Omega |\nabla w|^2 \, dx \geq \alpha \int_\Omega |w|^2 \, dx, \qquad \forall\, w \in H_0^1(\Omega).$$

Using this fact we obtain

$$a[v, v] = \nu \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx = \frac{\nu}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx + \frac{\nu}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx \geq$$

$$\geq \frac{\alpha \, \nu}{2} \int_{\Omega_\varepsilon} |v|^2 \, dx + \frac{\nu}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx \geq M \|v\|^2_{V(\Omega_\varepsilon)},$$

where $M = \min\{\frac{\alpha\,\nu}{2}, \frac{\nu}{2}\}$. Thus $a[\cdot, \cdot]$ is coercive. Finally, we need only to prove that f is linear and bounded. Using the same arguments as above, we conclude that $< f, \varphi >$ is linear. Furthermore, $< f, \varphi >$ is bounded since

$$< f, \varphi > \leq | < f, \varphi > | \leq \|f\|_{L^2(\Omega_\varepsilon)} \|\varphi\|_{V(\Omega_\varepsilon)} \leq C \|\varphi\|_{V(\Omega_\varepsilon)}.$$

So, we conclude that there exists a unique $v \in V(\Omega)$ in the sense of (7.2) which we refer as the *weak form* of (6.4) which often is presented as:

$$Find \; v \in V(\Omega_\varepsilon) \; such \; that \; a[v, \varphi] =< f, \varphi >, \qquad \forall\, \varphi \in V(\Omega_\varepsilon). \tag{7.4}$$

□

We obviously lost information in transforming (6.4) into the problem (7.4) since *the pressure term q is lost*. However, we will prove that there exist a unique pressure $q \in L^2_{loc}(\Omega_\varepsilon)$.

7.2 Existence and Uniqueness of the Pressure q

To be able to prove the existence and uniqueness of the pressure q in Ω_ε, we need to consult the theory of *calculus of variations*. This theory is commonly used in nonlinear problems where relative simple techniques from nonlinear functional analysis can be applied. The class of problems that is of interest in this case is referred as *variational problems*. In this technique, we consider the abstract nonlinear PDE

$$\mathcal{A}(v) = 0. \tag{7.5}$$

The nonlinear operator $\mathcal{A}(\cdot)$ is viewed as a derivative of an appropriate energy functional $\mathcal{J}(\cdot)$. Symbolically we write

$$\mathcal{A}(\cdot) = \mathcal{J}'(\cdot). \tag{7.6}$$

Then problem (7.5) reads

$$\mathcal{J}'(v) = 0. \tag{7.7}$$

The advantage of this new formulation is that we can know recognize solutions to (7.5) as being critical points of $\mathcal{J}(\cdot)$. These in certain circumstances may be relatively easy to find: if, for instance, the functional $\mathcal{J}(\cdot)$ has a minimum at v, then presumably (7.6) is valid and thus v is a solution to the original problem (7.5). For example, the weak form (7.4) can be viewed as a the minimization problem

$$\mathcal{J}(v) = \min_{\varphi \in V(\Omega_\varepsilon)} \mathcal{J}(\varphi), \tag{7.8}$$

where $\mathcal{J}(\varphi) = \frac{\nu}{2}\int_{\Omega_\varepsilon}(|\nabla\varphi|^2 - f\ \varphi)dx$. This is a consequence of the Fredrich's first inequality which then turned (7.4) into finding M to be able to secure coercivity of the functional $a[\cdot,\cdot]$. Therefore, (7.8) will satisfy (7.7). For more details of this theory we strongly recommend [3].

With this short introduction to the field of variational problems, we present the theorem that is needed to prove that there exist a unique pressure q. It turns out that we can only determine the uniqueness of q up to a constant.

Theorem 7.2.1 (Pressure as Lagrange multiplier). *There exist a scalar function $q \in L^2_{loc}(\Omega_\varepsilon)$ such that*

$$\int_{\Omega_\varepsilon} \nabla v\ \nabla\varphi\ dx - \int_{\Omega_\varepsilon} q\ \text{div}\ \varphi\ dx = \int_{\Omega_\varepsilon} f\ \varphi\ dx \tag{7.9}$$

for all $\varphi \in H^1(\Omega_\varepsilon)$ with compact support within Ω_ε.

Remark 7.2.2. *We interpret (7.9) as saying that $\{v, q\}$ form a weak solution of the Stokes' problem (6.4). The pressure q arises as a Lagrange multiplier corresponding to the incompressibility condition* div $v = 0$. *Since q is a scalar function, q is only unique up to a constant.*

Proof. See [3]. □

Chapter 8

Homogenization Process in Ω_1

In the above proofs we view v and q as functions dependent of one variable. To be able to detect the microscopic scale we need to define a microscopic equation. To do this we need to perform a homogenization process in Ω_1. We start by making a *multi scale expansion* in both v and q. Hence, we make the following *ansatz*:

$$v_\varepsilon = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + \dots \quad , y = \frac{x}{\varepsilon} \tag{8.1}$$

$$q_\varepsilon = q_0(x,y) + \varepsilon q_1(x,y) + \varepsilon^2 q_2(x,y) + \dots \quad , y = \frac{x}{\varepsilon} \tag{8.2}$$

where $v_j(x,y), q_j(x,y) \in L^2(\Omega_\varepsilon; H^1_{per}(\mathbf{Y}))$, for $j = 0, 1, \dots$. We choose both v_ε and q_ε to be periodic in $\mathbf{Y}$ since the rough boundary is periodic and we can by this fact use the theory of two-scale convergence. In other words, we can with this expansion investigate, asymptotically, how the flow is behaving in the periodic rough surface in the local variable y in the limit process.

Having defined the expansions (8.1) and (8.2) we now describe the flow in two different scales and thus we must transform the derivatives in the same manner. We then have by the chain rule, the *total derivatives*

$$\nabla_\varepsilon = \nabla_x + \frac{1}{\varepsilon}\,\nabla_y \tag{8.3}$$

$$\mathrm{div}_\varepsilon = \mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y \tag{8.4}$$

$$\Delta_\varepsilon = \Delta_x + \frac{2}{\varepsilon}\,\mathrm{div}_x(\nabla_y) + \frac{1}{\varepsilon^2}\Delta_y. \tag{8.5}$$

With (8.1)-(8.5) defined, we can establish a priori estimates of v_ε and q_ε in the sense of the both scales x and y. We start with estimating v_ε.

8.1 A priori estimate of v_ε

We have the following *energy estimate* in v_ε by simply replacing φ and v with v_ε in (7.3)

$$a[v_\varepsilon, v_\varepsilon] = \nu \|\nabla_\varepsilon v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega_\varepsilon)} \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{8.6}$$

In the above estimate, we do not use the fact that v_ε is a function of both x and y. To get hold of this, we need the following lemma, which can be viewed as a Poncaré inequality.

Lemma 8.1.1. *Let* $W(\Omega_\varepsilon) = \{w \in H^1(\Omega_\varepsilon) : w = 0 \text{ on } \partial\Omega_\varepsilon \setminus Z_- \varphi \text{ is } L-\text{periodic}\}$. *Then*

$$\int_{\Omega_1} |w|^2 \, dx \leq \frac{\varepsilon^2}{\alpha} \int_{\Omega_1} |\nabla_x \, w|^2 \, dx, \qquad \forall w \in W(\Omega_\varepsilon).$$

Proof. If we apply corollary (7.1.1) over $\mathbf{Y_F}$ for $w \in H^1(\mathbf{Y_F}) : w = 0$ on $\partial\mathbf{Y_F} \setminus \partial\mathbf{Y}$ we obtain

$$\int_{\mathbf{Y_F}} |w|^2 \, dy \leq \frac{1}{\alpha'} \int_{\mathbf{Y_F}} |\nabla_y \, w|^2 \, dy.$$

Change of variables $x = \varepsilon y$ yields $\nabla_x = \varepsilon \nabla_y$ and $\mathbf{Y_F} = \varepsilon \mathbf{Y_F}$. Thus

$$\int_{\varepsilon\mathbf{Y_F}} |w|^2 \, dx \leq \frac{\varepsilon^2}{\alpha'} \int_{\varepsilon\mathbf{Y_F}} |\nabla_x \, w|^2 \, dx.$$

Adding all integrals over Ω_1 and define $\alpha = \sum_{i=1}^{N(\varepsilon)} \alpha_i'$ we obtain lemma 8.1.1. □

If we apply the above lemma to the a priori estimate (8.6), we obtain the following estimates

$$\frac{1}{\varepsilon^2} \, \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C, \tag{8.7}$$

$$\frac{1}{\varepsilon} \, \|\nabla_\varepsilon v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C. \tag{8.8}$$

In (8.7) we see that the solution will blow up in the limit process as $\varepsilon \to 0$. Thus we need to scale our multi scaled expanded v_ε with a factor ε^2 to avoid this blow up. Note that this scale parameter dos not inflict on the existence and uniqueness of (6.4), since this holds for any order of magnitude of scale parameter $\varepsilon > 0$.

It is well known that the L^p-norm is preserved when extended by zero (the velocity obeys the no-slip condition on the rough surface) and thus is (8.7)-(8.8) valid in the whole Ω_1, i.e.,

$$\frac{1}{\varepsilon^2} \, \|v_\varepsilon\|_{L^2(\Omega_1)} \leq C,$$

$$\frac{1}{\varepsilon} \, \|\nabla_\varepsilon v_\varepsilon\|_{L^2(\Omega_1)} \leq C.$$

8.2 A priori estimate for q_ε

In this section we present the a priori estimates for the pressure q_ε that are established in [13]. The extension of the estimates in Ω_ε into Ω_1 is not trivial as in the case for the velocity. This is obvious since we can of course define a pressure over the rough (solid) part in Ω_1. If we define the extension of q_ε as q_ε' in the same way as [13], we obtain the following a priori estimates

$$\|q'_\varepsilon - \frac{1}{\Omega_1}\int_{\Omega_1} q'_\varepsilon \, dx\|_{L^2(\Omega_1)} \leq C, \tag{8.9}$$

$$\|\nabla_\varepsilon \, q'_\varepsilon\|_{W*} \leq C, \tag{8.10}$$

where q'_ε is defined as

$$q'_\varepsilon = \begin{cases} q_\varepsilon & \text{in } \Omega_\varepsilon \\ \frac{1}{\mathbf{Y}^\varepsilon_{\mathbf{S_i}}} \int_{\mathbf{Y}^\varepsilon_{\mathbf{S_i}}} q_\varepsilon \, dy & \text{in } \mathbf{Y}^\varepsilon_{\mathbf{S_i}} \text{ for each } i, \end{cases} \tag{8.11}$$

and $W = \{w \in H^1_{per}(\Omega_1) : \int_{\Omega_1} w \, dx = 0\}$. If we recall corollary (3.2.10) we conclude that the sequences $\{q'_\varepsilon - \frac{1}{\Omega_1}\int_{\Omega_1} q'_\varepsilon \, dx\}$ and $\{\nabla_\varepsilon \, q'_\varepsilon\}$ are strongly relatively compact in W^* and L^2, respectively.

8.3 Scaled Stokes' Equation

As a consequence of the a priori estimate (8.7), we define the *scaled Stokes' equation* as

$$\begin{cases} \mathcal{A}_\varepsilon := -\Delta_\varepsilon(\varepsilon^2 \, v_\varepsilon) + \nabla_\varepsilon q_\varepsilon = f(x) & \text{in } \Omega_1 \times \mathbf{Y_F} \\ \text{div}_\varepsilon \, (\varepsilon^2 \, v_\varepsilon) = 0 & \text{in } \Omega_1 \times \mathbf{Y_F} \\ \varepsilon^2 \, v_\varepsilon = 0 & \text{on } \Omega_1 \times \partial\mathbf{Y_F} \setminus \partial\mathbf{Y} \\ \{v_\varepsilon, q_\varepsilon\} \quad \text{is } 1-\text{periodic in } y, \end{cases} \tag{8.12}$$

where we have set $\nu = 1$ for simplicity. Note that the right hand side $f(x)$ is only a function of the macroscopic variable x, which write explicitly. If a function depends on both x and y we do not write it explicitly. The no-slip condition is slightly changed to fit the local variable y, but the physical interpretation is exactly the same as in (6.4). Having defined (8.12), we can start our asymptotical expansion. It is convenient to rewrite the operator $\mathcal{A}_\varepsilon$ to identify it as a function of power in ε as

$$\mathcal{A}_\varepsilon = \frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}(\mathcal{A}_1 + \mathcal{A}'_1) + (\mathcal{A}_2 + \mathcal{A}'_2),$$

where

$$\mathcal{A}_0 := -\Delta_y \tag{8.13}$$

$$\mathcal{A}_1 := -2 \, \text{div}_x \, \nabla_y \tag{8.14}$$

$$\mathcal{A}'_1 := \nabla_y \tag{8.15}$$

$$\mathcal{A}_2 := -\Delta_x \tag{8.16}$$

$$\mathcal{A}'_2 := \nabla_x. \tag{8.17}$$

Applying (8.13)-(8.17) we can identify the operator $\mathcal{A}_\varepsilon$ in (8.12) as

$$\mathcal{A}_\varepsilon = \frac{1}{\varepsilon}\mathcal{A}_1' q_0 + (\mathcal{A}_0 v_0 + \mathcal{A}_1' q_1 + \mathcal{A}_2' q_0) + \tag{8.18}$$

$$+\varepsilon(\mathcal{A}_0 v_1 + \mathcal{A}_1 v_0 + \mathcal{A}_1' q_2 + \mathcal{A}_2' q_1) +$$

$$+\varepsilon^2(\mathcal{A}_0 v_2 + \mathcal{A}_1 v_1 + \mathcal{A}_2 v_0 + \mathcal{A}_1' q_3 + \mathcal{A}_2' q_2) + \mathcal{O}(\varepsilon^3) = f(x).$$

To make use of the multi scale expansion, we begin to rewrite (8.18) in the different powers of ε with the corresponding boundary conditions. Note that for every $\mathcal{O}(\varepsilon^N)$, for $N = -1, 0, 1, \ldots$ $\{u_\varepsilon, p_\varepsilon\}$ is 1-periodic in y by the definition (8.1) and (8.2). Thus, we obtain the cascade equations

$\mathcal{O}(\varepsilon^{-1})$:

$$\mathcal{A}_1' q_0 = 0 \quad \text{in } \Omega_1 \times \mathbf{Y_F} \tag{8.19}$$

$\mathcal{O}(1)$:

$$\mathcal{A}_0 v_0 + \mathcal{A}_1' q_1 + \mathcal{A}_2' q_0 = f(x) \quad \text{in } \Omega_1 \times \mathbf{Y_F} \tag{8.20}$$

$\mathcal{O}(\varepsilon^1)$:

$$\begin{aligned} \mathcal{A}_0 v_1 + \mathcal{A}_1 v_0 + \mathcal{A}_1' q_2 + \mathcal{A}_2' q_1 = 0 \quad & \text{in } \Omega_1 \times \mathbf{Y_F} \\ \operatorname{div}_y v_0 = 0 \quad & \text{in } \Omega_1 \times \mathbf{Y_F} \end{aligned} \tag{8.21}$$

$\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned} \mathcal{A}_0 v_2 + \mathcal{A}_1 v_1 + \mathcal{A}_2 v_0 + \mathcal{A}_1' q_3 + \mathcal{A}_2' q_2 = 0 \quad & \text{in } \Omega_1 \times \mathbf{Y_F} \\ \operatorname{div}_x v_0 + \operatorname{div}_y v_1 = 0 \quad & \text{in } \Omega_1 \times \mathbf{Y_F} \\ v_0 = 0 \quad & \text{in } \Omega_1 \times \partial\mathbf{Y_F} \setminus \mathbf{Y}. \end{aligned} \tag{8.22}$$

As we mentioned in chapter 2, we are only interested in the lowest order of approximation, the so called homogenized problem. In the limit process as $\varepsilon \to 0$ the only term that survive is $\mathcal{O}(1)$. This implies that only equation (8.20) will determine our homogenized equation. We also need appropriate boundary conditions for the homogenized equation in order to find a unique solution. From the cascade equations (8.19)-(8.22), we use the boundary conditions prescribed for the velocity and the pressure in order to find the proper homogenized equation.

We see that, in view of (8.19),

$$\mathcal{A}_1' q_0 = \nabla_y q_0 = 0 \Leftrightarrow q_0 = q_0(x). \tag{8.23}$$

Also, the first boundary condition in (8.22) can be simplified. We recall the surjectivity of the div operator in the following lemma.

Lemma 8.3.1. *Let $G \in L^2(\mathbf{Y_F})$. Then there exist a $w \in H^1_{per}(\mathbf{Y_F})$ such that*

$$\begin{cases} \operatorname{div}_y w = G & \text{in } \mathbf{Y_F} \\ w = 0 & \text{on } \partial\mathbf{Y_F} \setminus \mathbf{Y} \end{cases} \tag{8.24}$$

if and only if

$$\int_{\mathbf{Y_F}} G \, dy = 0.$$

Since $v_0, v_1 \in H^1_{per}(\mathbf{Y_F})$ we conclude that, by lemma (8.3.1),

$$\begin{cases} \text{div}_y\ v_1 = -\text{div}_x\ v_0 & \text{in } \mathbf{Y_F} \\ v_1 = 0 & \text{on } \partial\mathbf{Y_F} \setminus \mathbf{Y} \end{cases} \tag{8.25}$$

if and only if

$$\text{div}_x \int_{\mathbf{Y_F}} v_0\ dy := \text{div}_x\ \bar{v}_0 = 0.$$

If we summarize the conditions for $\{v_0, q_0(x), q_1\}$ and let $\nu \neq 1$ we end up with the homogenized equation:

$$\begin{cases} -\nu\Delta_y\ v_0 + \nabla_y\ q_1 = f(x) - \nabla_x\ q_0(x) & \text{in } \Omega_1 \times \mathbf{Y_F} \\ \text{div}_y\ v_0 = 0 & \text{in } \Omega_1 \times \mathbf{Y_F} \\ \text{div}_x\ \bar{v}_0(x) = 0 & \text{in } \Omega_1 \times \mathbf{Y_F} \\ v_0 = 0 & \text{on } \partial\mathbf{Y_F} \setminus \partial\mathbf{Y} \\ \{v_0, p_1\} \quad \text{is } 1-\text{periodic in } y. & \end{cases} \tag{8.26}$$

We refer (8.26) as the homogenized equation of (6.4). In order to find a unique solution in an appropriate functional space we need to impose more boundary conditions in the x-variable. It turns out that, after a Fredholm alternative argument, if we add

$$\{p_0(x), \int_{\mathbf{Y_F}} u_0\ dy = 0\} \quad \text{is } L-\text{periodic}, \tag{8.27}$$

to (8.26), we can find unique solutions to $\{u_0, p_0(x), p_1\}$. The system (8.26)-(8.27) is referred to as the Stoke's system with two pressures. To find a variational form of the system (8.26)-(8.27), we introduce the following Hilbert spaces

$$H^\sharp(\mathbf{Y_F}) = \{\varphi \in H^1_{per}(\mathbf{Y_F}) : \text{div}_y\ \varphi = 0 \text{ in } \mathbf{Y_F}\ : \varphi = 0 \text{ on } \partial\mathbf{Y_F} \setminus \partial\mathbf{Y}\}$$

$$W^\sharp = \{w \in L^2(\Omega_1; H^\sharp(\mathbf{Y_F})) : \text{div}_x \int_{\mathbf{Y_F}} w(x,y)\ dy = 0 \text{ in } \Omega_1$$

$$n(x) \int_{\mathbf{Y_F}} w(x,y)\ dy \text{ is } H^{-1/2}\text{-antiperiodic}, \}$$

where $n(x)$ is the outward unit normal on Ω_1. Also we need the corresponding bilinear forms

$$a_{\mathbf{Y_F}}[w, \varphi] = \int_{\mathbf{Y_F}} \nabla_y\ w\ \nabla_y\ \varphi\ dy, \qquad w, \varphi \in H^\sharp(\mathbf{Y_F}) \tag{8.28}$$

$$a_{\Omega_1}[w, \varphi] = \int_{\Omega_1} a_{\mathbf{Y_F}}[w, \varphi]\ dx, \qquad w, \varphi \in W^\sharp \tag{8.29}$$

$$< f, \varphi >_{\Omega_1 \times \mathbf{Y_F}} = \int_{\Omega_1} \int_{\mathbf{Y_F}} f\ \varphi\ dydx, \qquad f \in W^{\sharp *}, \varphi \in W^\sharp. \tag{8.30}$$

With these Hilbert spaces we can define a variational form of the system (8.26)-(8.27) . We thus end up with the following variational formulation

$$Find\ v_0 \in W^\sharp\ such\ that\ a_{\Omega_1}[v_0, \varphi] =< f, \varphi >_{\Omega_1 \times \mathbf{Y_F}}, \qquad \forall\ \varphi \in W^\sharp. \tag{8.31}$$

Theorem 8.3.2. *There exist a unique $v_0 \in W^\sharp$ such that (8.31) holds for all $\varphi \in W^\sharp$.*

Proof. Obviously, $a_{\Omega_1}[v_0, \varphi]$ defines an inner product and a norm on $W^\sharp$. Moreover, $W^\sharp$ is a Hilbert space with respect to the inner product induced by a_{Ω_1}. Consequently, the Lax-Milgram theorem implies a unique $v_0 \in W^\sharp$ in the sense of (8.31). □

As in the first variational formulation (7.4), the pressure terms are eliminated. We thus need to find the unique pressure terms $q_0(x)$ and q_1.

Theorem 8.3.3. *Let v_0 be a solution to (8.31). Then there exist a unique $q_0(x) \in H^1_{per},\ \int_{\Omega_1} q_0(x)\ dx = 0$ and a unique $q_1 \in L^2(\Omega_1 \times \mathbf{Y_F}),\ \int_{\Omega_1}\int_{\mathbf{Y_F}} q_1\ dydx = 0$ such that (8.26)-(8.27) holds in the sense of distributions.*

Proof. We are going to show that an interpretation of (8.31) gives $q_0(x)$ and q_1, as the interpretation of a weak solution of the Stokes' system which then gives the pressure field. Let $\Upsilon = \{\varphi \in C^\infty_{per}(\Omega_1) : \text{div } \varphi = 0 \text{ in } \Omega_1\}$ and $\mathcal{H}$ be the completion of Υ in Ω_1 with respect to the L^2-norm. Then $\mathcal{H} = \{z \in L^2(\Omega) : \text{div } z = 0,\ nz\ is\ L-\text{antiperiodic}\}$ and $\mathcal{H}^\perp = \{z \in L^2(\Omega_1) :\ z = \nabla\ v,\ v \in H^1_{per}(\Omega_1)\}$.

If we consider the Stokes' system with the two pressures, i.e., system (8.26)-(8.27), we separate the variables as

$$v_0 = \frac{1}{\nu}\sum_{i=1}^{2}(f(x) - \nabla_x p_0(x))_i\ \omega_i(y), \quad x \in \Omega_1,\ y \in \mathbf{Y_F} \tag{8.32}$$

$$q_1 = \sum_{i=1}^{2}(f(x) - \nabla_x q_0(x))_i\ \pi_i(y), \quad x \in \Omega_1,\ y \in \mathbf{Y_F}. \tag{8.33}$$

Substitute (8.32)-(8.33) into (8.26)-(8.27) and divide by $(f(x) - \nabla_x p_0(x))_i$ we obtain the *cell problem* in $\mathbf{Y_F}$:

For $i = 1, 2$, find $\{\omega_i(y), \pi_i(y)\} \in H^1_{per}(\mathbf{Y_F}) \times L^2(\mathbf{Y_F})$ such that

$$\begin{cases} -\Delta_y\ \omega_i(y) + \nabla_y\ \pi_i(y) = e_i & \text{in } \mathbf{Y_F} \\ \text{div}_y\ \omega_i(y) = 0 & \text{in } \mathbf{Y_F} \\ \omega_i(y) = 0 & \text{on } \partial\mathbf{Y_F} \setminus \partial\mathbf{Y} \\ \int_{\mathbf{Y_F}} \omega_i(y)\ dy = 0, & \end{cases} \tag{8.34}$$

where e_i represents the unit vector. Again, there exist a unique solution $\omega_i(y)$ to the cell problem (8.34) by the Lax-Milgram theorem. Moreover, there exist a unique solution $\pi_i(y)$ to (8.34) by the De Rahms' theorem. We define the *permeability matrix* K as

$$K_{ij} = \int_{\mathbf{Y_F}} \omega_{ij}\ dy \quad , 1 \le i, j \le 2$$

The permeability matrix is symmetric and positive definite. Consequently, the *drag matrix* K^{-1} is then positive definite. Now, let $\theta \in L^2(\Omega_1)$. Following [13] we set Φ to be the scalar product $\Phi = \sum_{i=1}^{2}(K^{-1}, e_i)\ \omega_i(y)$. Then $\Phi \in L^2(\Omega_1 \times H^\sharp(\mathbf{Y_F}))$ and satisfies

$$\int_{\mathbf{Y_F}} \Phi \, dy = \theta(x) \quad \text{in } \Omega_1 \quad \text{and} \quad \|\Phi\|_{L^2(\Omega_1 \times H^\sharp(\mathbf{Y_F}))} \leq C\|\theta\|_{L^2(\Omega_1)}.$$

Hence, the mean value θ of Φ over $\mathbf{Y_F}$ is a bounded surjective operator between $L^2(\Omega_1 \times H^\sharp(\mathbf{Y_F}))$ and $L^2(\Omega_1)$. Now, let $\theta \in \mathcal{H}$. Then there exist a $\varphi_\theta \in W^\sharp$ such that $\int_{\mathbf{Y_F}} \varphi_\theta \, dy = \theta$ and

$$a_{\Omega_1}[v_0, \varphi_\theta] =< f, \varphi_\theta >_{(\Omega_1 \times H^\sharp(\mathbf{Y_F}))} = \int_{\Omega_1} f(x) \left(\int_{\mathbf{Y_F}} \varphi_\theta \, dy \right) \, dx = \int_{\Omega_1} f(x)\theta(x) dx.$$

Therefore, there is a $q_0(x) \in H^1_{per}(\Omega_1)$, $\int_{\Omega_1} q_0(x) \, dx = 0$ such that $\forall \theta \in L^2(\Omega_1)$ we have

$$a_{\Omega_1}[v_0, \varphi_\theta] = \int_{\Omega_1} (f(x) - \nabla_x q_0(x))(\int_{\mathbf{Y_F}} \varphi_\theta \, dy) \, dx = \tag{8.35}$$

$$= \int_{\Omega_1} (f(x) - \nabla_x q_0(x))\theta(x) \, dx.$$

Condition $\int_{\Omega_1} q_0(x) \, dx = 0$ implies uniqueness of $q_0(x)$. (8.35) is equivalent to

$$a_{\mathbf{Y_F}}[v_0(x, \cdot), \varphi] = (f(x) - \nabla_x q_0(x)) \int_{\mathbf{Y_F}} \varphi \, dy, \tag{8.36}$$

$\forall \varphi \in H^\sharp(\mathbf{Y_F})$ a.e. on Ω_1. Existence and uniqueness of $q_1 \in L^2(\Omega \times \mathbf{Y_F})$, $\int_{\mathbf{Y_F}} \int_{\Omega_1} q_1 \, dxdy = 0$, such that (8.26)-(8.27) holds in the sense of distributions, is now a classical result an we refer to the classical textbook [19]. □

Remark 8.3.4 (Darcy's law). *If we integrate to obtain the mean velocity over the cell then $\bar{v}_0(x)$ obeys the Darcy's law, i.e.,*

$$\bar{v}_0(x) = \int_{\mathbf{Y_F}} v_0 \, dy = \frac{K}{\nu}(f(x) - \nabla_x q_0(x)), \tag{8.37}$$

where K is the permeability matrix.

Remark 8.3.5. *Darcy's law is sometimes written as*

$$\text{div } \bar{v}_0(x) = 0. \tag{8.38}$$

We recognize $\bar{v}_0(x)$ as the *seepage velocity*. Hence, we can identify the rough surface as a porous medium since the average velocity is the seepage velocity divided by $|\mathbf{Y_F}|$ which is the porosity of Ω_1. This is indeed a very important observation from a physical point of view since the rough surface asymptotically tends to a porous medium. The cell problem (8.34) is then the equation in $\Omega_1 \setminus Z_-$ which describes the flow characteristics at the microscopic scale.

To be able to use the cell problem in such way that we can find a corresponding macroscopic law that describes the microscopic behaviour, we need not only the cell problem but also the equations that are valid in Z_{bl} and in Ω_2. We proceed in chapter 10 with the equations governing Ω_2.

Chapter 9

Convergence Result

In this chapter we examine the two-scale convergence in L^2 of the sequence $\{v_\varepsilon\}$ towards v_0 and strong L^2-convergence of q' towards q_0. For simplicity, we choose $\int_{\Omega_\varepsilon} q_\varepsilon \, dxdy = 0$

9.1 Two-scale Convergence in the Homogenization Process

Theorem 9.1.1. *Let $\{v_\varepsilon, q_\varepsilon\}$ be solutions to (8.12) and $\int_{\Omega_\varepsilon} q_\varepsilon \, dxdy = 0$. We suppose v_ε is extended by zero in Ω_1 and the extension q'_ε is given by (8.11). Let v_0 and q_0 be given by (8.26). Then*

$$\frac{1}{\varepsilon^2} \, v_\varepsilon \overset{2}{\rightharpoonup} v_0 \quad in \;\; \Omega_1 \tag{9.1}$$

$$\frac{1}{\varepsilon} \, \nabla_\varepsilon \, v_\varepsilon \overset{2}{\rightharpoonup} \nabla_y \, v_0 \quad in \;\; \Omega_1 \tag{9.2}$$

$$q'_\varepsilon \to q_0(x) \quad in \;\; L^2(\Omega_1) \tag{9.3}$$

Proof. By the multi scale expansion and the scale parameter introduced in the Stokes' system we have for $\varphi\left(x, \frac{x}{\varepsilon}\right) := \varphi_\varepsilon \in L^2(\Omega; H^1_{per}(\mathbf{Y_F}))$

$$\lim_{\varepsilon \to 0} \int_{\Omega_1} \Delta_\varepsilon(\varepsilon^2 \, v_\varepsilon) \, \varphi_\varepsilon dx = \int_{\Omega_1} \int_{\mathbf{Y_F}} \Delta_y \, \widetilde{v}_0(x,y) \varphi(x,y) \, dydx \tag{9.4}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Omega_1} \nabla_\varepsilon(\varepsilon^2 \, v_\varepsilon) \, \varphi_\varepsilon dx = \int_{\Omega_1} \int_{\mathbf{Y_F}} \nabla_y \, \widetilde{v}_0(x,y) \varphi(x,y) \, dydx. \tag{9.5}$$

We need to prove that $\widetilde{v}_0 = v_0$ in order to obtain (9.1)-(9.2). For this proof see [13] as well as the proof for (9.3). □

Chapter 10

Equation for the Rough Surface of height ε

With the homogenization process given in $\Omega_1 \setminus Z_-$, we need to find the proper equation that describes the flow in $\Omega_2 \cup Z_-$, i.e., the one for u_ε. In order to understand how the flow behaves near a body, we need to understand the fundamental theory of boundary-layers. We continue to explain the basic concepts of boundary-layer theory with the aid of purely physical ideas.

10.1 Basic Boundary-Layers Theory

The boundary-layer concept was introduced by L. Prandtl in 1904. In his hypothesis the fluid friction on a body is limited to a thin layer near the boundary of a body, hence the term boundary-layer. He proposed that the boundary-layer thickness δ for a flow over a plate is the distance away from the the surface of the plate where the velocity reaches $0.99\ U_f$, where U_f free stream velocity.

This theory was further developed by H. Schlichting. According Schlichting, the boundary layer is divided in to three regions. In theses three regions, different equations, or rather velocity profiles, describes the flow. To separate these regions, a dimensionless number $\eta_+ = \frac{x_2}{\delta}$ is introduced. η_+ is interpreted as a characteristic wall coordinate.

The *viscous sublayer* is the region nearest the plate where $\eta_+ \leq 5$ and the velocity profile in this region is consider to be linear. For $5 < \eta_+ < 20$ the region is called the *buffer layer*. Finally, the *logarithmic sublayer* is defined for $\eta_+ > 20$ where the profile can be described by a logarithmic law.

10.2 The Viscous Sublayer

In this thesis we only consider the flow within the viscous sublayer. Hence, the equation that describes the flow in $\Omega_2 \cup Z_-$ is a linear velocity profile. It is well known that this kind of flow is described by the Couette flow, which is a shear-driven fluid motion with no-slip on the wall. This equation is given by

$$\begin{cases} (u_\varepsilon \cdot \nabla)u_\varepsilon - \nu\Delta u_\varepsilon + \nabla p_\varepsilon = 0 & \text{in } \Omega_2 \cup Z_- \\ \text{div } u_\varepsilon = 0 & \text{in } \Omega_2 \cup Z_- \\ u_\varepsilon = 0 & \text{on } \partial(\Omega_2 \cup Z_-) \\ u_\varepsilon = (U, 0) & \text{on } (\cdot, L_2') \\ \{u_\varepsilon, p_\varepsilon\} & \text{is } x_1 - \text{periodic.} \end{cases} \tag{10.1}$$

where $u_\varepsilon = u_\varepsilon(x, y)$ is the velocity field, $p_\varepsilon = p_\varepsilon(x, y)$ is the pressure and $L_2' >> \varepsilon$ the characteristic height of $\Omega_2 \cup Z_-$. It is proved in [8] that (10.1) admits a unique solution $\{u_\varepsilon, p_\varepsilon\} \in H^2(\Omega_2) \times H^1(\Omega_2)$ for

$$Re = \frac{|U|L_2'}{\nu} < 2, \tag{10.2}$$

where Re is the dimensionless Reynolds number. The Reynolds number is the ratio between the inertial forces and the viscous forces. It is very important to understand that *the Reynolds number is a property of the flow, not the fluid.* In general, for high Reynolds number the flow is considered turbulent where the inertia forces are dominant. On the other hand, for low Reynolds number the flow is laminar and the flow is governed by viscous forces, as in the case for the viscous sublayer. Moreover, the thickness of the boundary-layer is given by

$$\delta \sim \sqrt{\frac{\nu L_2'}{U}}.$$

If we choose water as the fluid, then $\nu = 10^{-6}\ [\frac{m^2}{s}]$ at room temperature. Due to the condition (10.2), we set $U = 10^{-3}[\frac{m}{s}]$ and $L_2' = 10^{-3}[m]$ which yields $Re = 1 < 2$ and $\delta = 10^{-3}[m]$. Hence, the viscous sublayer is approximately $5 \cdot 10^{-3}[m]$. The Couette flow is then valid in this range for $Re < 2$.

To summarize, we now have a microscopic law, the cell problem (8.34), which describes the flow in $\Omega_1 \setminus Z_-$ and (10.1) in $\Omega_1 \cup Z_-$ which describes the Couette flow in both the macroscopic and microscopic scale. As mentioned in the introduction, to calculate the solution $\{u_\varepsilon, p_\varepsilon\}$ in (10.1) is numerically difficult. Because of the rough boundary we need a very fine mesh to capture the flow behaviour. As we will see in the next chapters, we can replace the rough surface with an artificial smooth surface Z_{bl}.

Chapter 11

The Boundary-Layer Equation

We here present the auxiliary boundary-layer equation that is valid in the boundary $Z_* = Z_- \cup Z_{bl} \cup Z_+$. This equation has been rigorously studied by Andro Mikelić and Willy Jäger in [5], [6], [7] and [9]. In these papers, the result from the homogenization process in $\Omega_1 \setminus Z_-$ has been used to find appropriate velocity corrector terms. With these corrector terms, an artificial smooth surface Z_{bl} is obtained which is an $\mathcal{O}(\varepsilon^2)$ approximation of (10.1).

11.1 The Auxiliary Boundary-Layer equation

The equation that is valid in Z_* is called the *auxiliary boundary-layer equation*,

$$\left\{\begin{array}{rl} -\Delta_y\, \omega_*(y) + \nabla_y\, \pi_*(y) = 0 & \text{in } Z_+ \cup Z_- \\ \text{div}_y\, \omega_*(y) = 0 & \text{in } Z_* \\ (\omega_*(y))_{Z_{bl}}(\cdot, 0) = 0 & \text{on } Z_{bl} \\ ((\nabla_y\, \omega_*(y) - \pi_*(y)\mathcal{I})e_2)_{Z_{bl}}(\cdot, 0) = e_1 & \text{on } Z_{bl} \\ \omega_*(y) = 0 & \text{on } \mathbf{S} \\ \{\omega_*, \pi_*\} \quad \text{is } x_1 - \text{periodic}, & \end{array}\right. \tag{11.1}$$

where ω_* is the velocity field and π_* is the pressure. This equation is quite similar with the cell problem (8.34) and in a sloppy sense can (11.1) be viewed as the homogenous cell problem with the additional constraint $((\nabla_y\, \omega_*(y) - \pi_*(y)\mathcal{I})e_2)_{Z_{bl}}(\cdot, 0) = e_1$ on the interface between Ω_1 and Ω_2.

Let $V_* = \{\varphi \in L^2_{loc}(Z_*) :\ \nabla_y\, \varphi \in L^2_{loc}(Z_*) :\ \varphi = 0 \text{ on } \mathbf{S} :\ \text{div}_y\, \varphi = 0 \text{ in } Z_*\}$. Using test functions φ from this space we get the following weak formulation of (11.1):

$$Find\ \omega_* \in V_*\ such\ that\ a_*[\omega_*, \varphi] =< e_1, \varphi >, \qquad \forall\, \varphi \in V_* \tag{11.2}$$

where

$$a_*[\omega_*, \varphi] = \int_{Z^*} \nabla_y\, \omega_*\, \nabla_y\, \varphi\, dy$$

and

$$< e_1, \varphi >= -\int_{Z_{bl}} e^1\, \varphi\, dy.$$

Theorem 11.1.1. *There exist a unique solution $\{\omega_*, \pi_*\}$ to (11.1).*

Proof. Using Lax-Milgram theorem there is a unique $\omega_* \in V_*$ satisfying (11.2) and using De Rahm's theorem we obtain $\pi_* \in L^2_{loc}(Z_*)$ satisfying (11.2), which is unique up to a constant. □

In the next lemma we recall two important properties of the solutions to the auxiliary boundary-layer equation.

Lemma 11.1.2 (Navier Constant C_*)**.** *Any solution* $\{\omega_*, \pi_*\}$ *of (11.1) satisfies*

$$C_* = \int_0^1 \omega_*(y_1, 0)\, dy_1 = -\int_{Z_*} |\nabla_y\, \omega_*|^2\, dy \tag{11.3}$$

$$C_*^a = \int_0^1 \omega_*(y_1, a)\, dy_1 = C_* - a \quad , h - 1 < a, \tag{11.4}$$

where h is the height of the rough surface. In the next chapter we will identify this constant C_* as the constant coefficient operator that will give the effective equation $u^e(x)$. Furthermore, we will also identify C_* in the effective equation as the Navier constant.

Chapter 12

The Effective Equation

As mentioned in the introduction we are interested in finding the effective equation $u^e(x)$ which approximates u_ε. Until this point, we have established the equation for u_ε in (10.1) and the equation for the rough surface in the boundary Z_* between Ω_1 and Ω_2 via the auxiliary boundary layer equation (11.1). In order to find the effective equation, we need to find the last equation, i.e. an equation for $u(x)$, which is the Couette flow in Ω_2. With this last equation, we can use all the information obtained in the homogenization process, which lead to the auxiliary boundary problem, and finally find $u^e(x)$. We shall see that in the homogenization process the rough boundary is replaced, asymptotically, with an artificial smooth boundary where the constant coefficient operator is the Navier constant C_*.

12.1 Couette Flow in Ω_2

Since we only consider the viscous sublayer, the flow in Ω_2 is governed by the Couette flow with the no-slip on the flat surface at $x_2 = 0$. The solution to the Couette flow is given by

$$u(x) = \frac{Ux_2}{L_2} \tag{12.1}$$

$$p(x) = 0, \tag{12.2}$$

where L_2 is the height of Ω_2.

12.2 Derivation of the Effective Equation

Having defined all equation that is needed, we now present the results that Andro Mikelić and Willy Jäger obtained in [7] and [8]. In these papers, the authors obtained uniform a priori estimates for approximation of u_ε. It is proved in these papers that the Couette flow u is an $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ approximation of u_ε in $L^2(\Omega_2)$:

$$u_\varepsilon = u - \varepsilon\, \omega_* \frac{\partial u_1}{\partial x_2} + \varepsilon C_* \left(\frac{\partial u_1}{\partial x_2} e_1 + d_1 \right) H(x_2) + \mathcal{O}(\varepsilon^2), \tag{12.3}$$

where $H(x_2)$ is the Heaviside distribution and d_1 corresponds to the counter flow generated by the artificial boundary Z_{bl}, which is , when restricted to Z_{bl}, an $\mathcal{O}(\varepsilon)$ of u_ε on Z_{bl} in $L^2(Z_{bl})$:

$$\frac{\partial u_{\varepsilon 1}}{\partial x_2} = \frac{\partial u_1}{\partial y_2}\left(1 - \frac{\partial \omega_{*1}}{\partial y_2}\right) + \mathcal{O}(\varepsilon)$$
$$\frac{1}{\varepsilon} u_{\varepsilon 1} = -\frac{\partial u_1}{\partial y_2}\omega_{*1} + \mathcal{O}(\varepsilon).$$

The first term on the right hand side of (12.3) is the Couette flow, the second is the first order boundary layer correction to obtain the correct boundary condition at the artificial smooth boundary Z_{bl}. This term is a constant order ε. This means that the linear profile is shifted to the right by this amount which gives the wrong boundary condition on the upper boundary where the velocity U is prescribed. To correct this we have to add the third term which contains the counter flow and the resulting profile is again a linear one with a lower origin.

After averaging these equations on Z_{bl} and neglecting higher order terms the effective equation is obtained on the artificial smooth surface as:

$$u^e = -\varepsilon C_* \frac{\partial u_i^e}{\partial x_2}. \tag{12.4}$$

With (12.4) defined we have derived, through a homogenization process, a macroscopic law that describes the flow over a rough surface of characteristic height ε which captures the flow behaviuor at the microscopic scale through the Navier constant C_*. The effective Couette Navier flow is then given by:

$$\begin{cases} (u^e \cdot \nabla)u^e - \nu\Delta u^e + \nabla p^e = 0 & \text{in } \Omega_2 \\ \text{div } u^e = 0 & \text{in } \Omega_2 \\ u^e = (U, 0) & \text{for } x_2 = L_2 \\ u_1^e = -\varepsilon C_* \frac{\partial u_i^e}{\partial x_2} & \text{on } Z_{bl} \\ u_2^e = 0 & \text{on } Z_{bl} \\ \{u^e, p^e\} & \text{is } x_1 - \text{periodic.} \end{cases} \tag{12.5}$$

Theorem 12.2.1. *If $Re = \frac{|U|L_2}{\nu} < 2$, there exist a unique solution to (12.5):*

$$\begin{cases} u^e = (U + (\frac{x_2}{L_2})(1 - \frac{\varepsilon}{L_2}C_*)^{-1}), & x \in \Omega_2 \\ p^e = 0, & x \in \Omega_2 \end{cases} \tag{12.6}$$

Proof. See [8]. □

Hence, we have replaced the equation (10.1) with (12.5). Clearly, if we choose $\varepsilon = 0$ we obtain the flow over the flat plate with the no-slip condition at the wall at $x_2 = 0$. The boundary condition on the artificial boundary is referred as the Navier slip condtion and it was suggested by C.L. Navier in the early 19th century. His argument was based on physics, which is quite different from the argument used in this thesis.

12.3 Drag Reduction Over the Rough Surface

Having derived (12.5) we can now investigate if the tangential drag force is reduced for the microscopic rough surface. If we only consider the skin friction the tangential drag force on Z_{bl} is defined as

$$\mathcal{F}_t(u_\varepsilon) = \frac{\nu}{L}\int_{Z_{bl}} n \cdot \sigma_\varepsilon \cdot e_1 \, dx_1, \tag{12.7}$$

where $n = e_2$ is the normal vector to Z_{bl} and $\sigma_{\varepsilon_{ij}} = \frac{1}{2}(\frac{\partial u_{\varepsilon i}}{\partial x_j} + \frac{\partial u_{\varepsilon j}}{\partial x_i}) - p_\varepsilon \delta_{ij}$ is the total stress tensor, consisting of both viscous shear stress and pressure stress. The corresponding effective drag is

$$\mathcal{F}_t(u^e) = \frac{\nu}{2L}\int_{Z_{bl}} \left(\frac{\partial}{\partial x_2}u_1^e(x_1,0) + \frac{\partial}{\partial x_1}u_2^e(x_1,0)\right) \, dx_1 = \frac{\nu}{2L}\int_{Z_{bl}} \frac{\partial}{\partial x_2}u_1^e(x_1,0) \, dx_1.$$

The last equation holds since the Couette flow is independent of x_1. If we insert the effective Couette flow (12.6) in (12.7), we get

$$\mathcal{F}_t(u^e) = \frac{\nu}{2} \, \frac{U}{L_2 - \varepsilon C_*}. \tag{12.8}$$

By the definition of C_*, we see that (12.8) reaches a maximum when $\varepsilon = 0$, i.e., when we have a flat plate. Hence, the tangential drag force is reduced when a microscopic rough surface is introduced in the viscous sublayer. It is important to emphasize that the approximate drag force is an $\mathcal{O}(\varepsilon^2)$ approximation and it is proved in [8] that

$$|\mathcal{F}_t(u_\varepsilon) - \mathcal{F}_t(u^e)| \leq \varepsilon^2 C_* \frac{U^2}{\nu L_2}(1 + \frac{\nu}{L_2 U}).$$

Chapter 13

Numerical Simulations of C_*

As a consequence of the effective equation the rough surface reduces the drag. By the definition (11.3), C_* is always negative. In order to maximize the drag reduction introduced by rough surface, the Navier constant C_* is then to be minimized. Clearly, C_* is then highly shape dependent since the problem can be formulated

$$\begin{cases} \min_{\gamma \in \mathcal{R}} \ C_* \\ \textit{s.t.} \ a_*[\omega_*, \varphi] =< e_1, \varphi >, \end{cases} \tag{13.1}$$

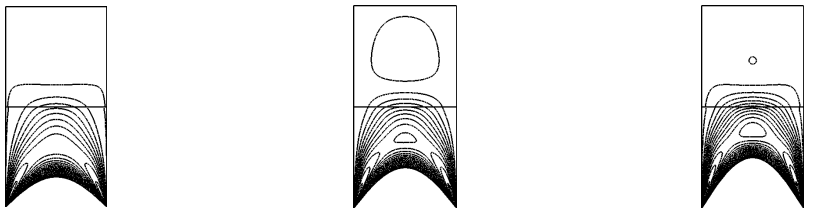
where γ is the shape of a rough structure $\mathcal{R}$. In [4] this shape optimization problem is studied in detail. In this section we only give numerical result with no intention of finding the optimal shape.

13.1 Simulations using Comsol Multiphysics

Comsol Multiphysics were used to simulate the auxiliary boundary problem in Z_* to obtain C_*^a. We have used the data that is presented in the section 10.2. The simulations where performed by a fixed $\varepsilon = 1 \times 10^{-4}[m]$ with the same shape but different hight h of the microstructure. The result can be found in the table below.

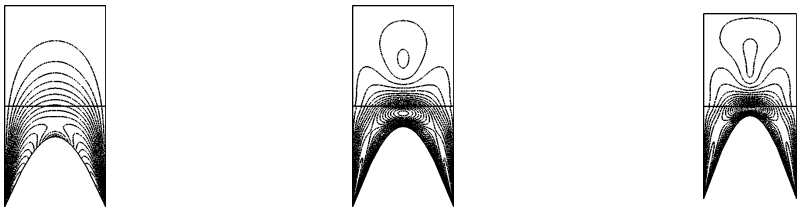
h	C_*^a
0.3	-0.6989
0.4	-0.5988
0.5	-0.4986
0.6	-0.3983
0.7	-0.2984
0.8	-0.1970
0.9	-0.0952

The Navier constant depends linearly on height for a fixed shape of the rough surface. This is expected since, for example, a small fish has a lower drag than a large fish with the same shape. The contour plots for the solution ω_* in Z_* are displayed for the different heights below.



(a) $h = 0.3$ (b) $h = 0.4$ (c) $h = 0.5$

Figure 13.1: The solution ω_* to the auxiliary boundary problem



(a) $h = 0.7$ (b) $h = 0.8$ (c) $h = 0.9$

Figure 13.2: The solution ω_* to the auxiliary boundary problem

Chapter 14

Conclusions

In essence, we have achieved our main goal to explain how the drag force can be reduced when a rough surface is introduced over a flat plate. The expression (12.8) clearly illustrates this fact.

We have also showed that, through a homogenization process in Ω_1, we can describe the microscopic behaviuor of the flow in the macroscopic scale where the dependence of the microscopic scale is captured in the constant coefficient operator C_*. This effective equation can be treated numerically once the coefficient operator C_* has been established. Moreover, the rough surface is replaced by an artificial smooth boundary in the effective equation, which is as an $\mathcal{O}(\varepsilon^2)$ approximation, with the Navier slip condition. A boundary condition that was suggested almost two centuries ago.

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