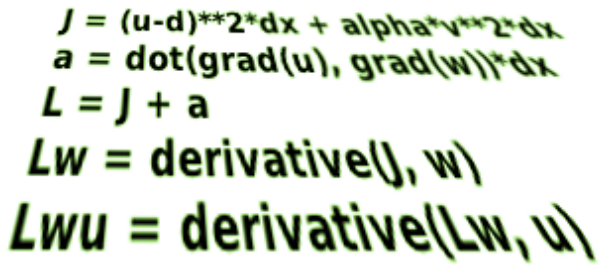
Numerical comparison of preconditioned oneshot and steepest descent on linear elliptic optimal control problems



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An optimal control problem constrained by linear elliptic PDE

Find $u, y = y(u)$ to minimize

$$J(u, y(u)) = \int_D (y(u) - z)^2 \, \mathrm{d}x + \alpha \int_\Omega u^2 \, \mathrm{d}x, \tag{1}$$

under the constraint of

$$(1 - \Delta)y = \chi u, \tag{2}$$

discretized using Galerkin FEM, in weak form

$$a(y, \phi) = b(u, \phi), \forall \phi \in U. \tag{3}$$

The preconditioned oneshot method

- Define the Lagrangian L with Langrange multiplier p: $L = J(u, y) + a(y, p) - b(u, p)$
- Differentiate L w.r.t. (y, u, p) once and twice to get the optimality system; $L''(y, u, p) = -L'$
- Assemble linear system as block matrices.
- Define block preconditioner with ML (AMG from Trilinos).
- Solve using MinRes.

Steepest descent method with optimal step length (1/2)

Select initial control u^0, solve $Ay^0 = Bu^0$ with B.C., and iterate:
Compute gradient of J, use as update vector for control:

$$A^* p^k = \frac{\partial J}{\partial y}, \tag{4}$$

$$d^k = \frac{\partial J}{\partial u} + B^* p^k, \tag{5}$$

Compute corresponding update direction for state:

$$Aq^k = Bd^k, \tag{6}$$

Then update iterates with some steplength τ^k:

$$u^{k+1} = u^k + \tau^k d^k, \tag{7}$$

$$y^{k+1} = y^k + \tau^k q^k. \tag{8}$$

Steepest descent method with optimal step length (2/2)

Choose τ^k to minimize the quadratic function

$$J(u^{k+1}, y^{k+1}) = J(\tau) = J(u^k + \tau d^k, y^k + \tau q^k). \tag{9}$$

The minimum is at

$$\tau^k = \frac{(y^k - z, q^k)_D + \alpha(u^k, d^k)}{(q^k, q^k)_D + \alpha(d^k, d^k)}. \tag{10}$$

Analytical solution to distributed control problem (1/2)

First choose an orthogonal basis:

$$N_j = \sin(j\pi x)\sin(j\pi y) \tag{11}$$

with the property

$$(1-\Delta)N_j = \lambda_j N_j, \qquad \lambda_j = (1+2(j\pi)^2), \tag{12}$$

then pick coefficients for a target state and manufacture the control

$$y = \sum_j c_j^y N_j, \tag{13}$$

$$u = (1-\Delta)y = \sum_j \lambda_j c_j^y N_j. \tag{14}$$

Analytical solution to distributed control problem (2/2)

By requiring $\frac{\partial J}{\partial c_j^u} = 0$, $\forall c_j^u$, after some calculations a matching observation z_α can be determined as

$$z_\alpha = \sum_j c_j^z N_j, \tag{15}$$

$$c_j^z = (1 + \alpha\lambda_j^2) c_j^y. \tag{16}$$

Note in particular that

$$\frac{c_j^z}{c_j^y} = 1 + \alpha\lambda_j^2 = O(\alpha j^4). \tag{17}$$

When using z_α in solver, $\|u - u_0\|/\|u_0\| \sim h^2$

n, α	1.0e+00	1.0e-02	1.0e-04	1.0e-06	1.0e-08
16	1.63e-01	1.63e-01	1.36e-01	1.59e-01	2.40e-01
32	4.46e-02	4.44e-02	3.56e-02	3.95e-02	6.56e-02
64	1.14e-02	1.13e-02	9.00e-03	9.82e-03	1.75e-02
128	2.86e-03	2.85e-03	2.26e-03	2.45e-03	4.44e-03
256	7.16e-04	7.14e-04	5.64e-04	6.12e-04	1.11e-03

The preconditioner keeps the number of minres iterations for the oneshot system ~constant

Setting initial guess to zero, running to manufactured solution with $\|r_{k^*}\| < 10^{-10}$.

n, α	1.0e+00	1.0e-02	1.0e-04	1.0e-06	1.0e-08
16	16	25	41	41	21
32	23	32	44	43	29
64	28	39	56	50	34
128	31	45	64	58	36
256	38	55	73	69	43

The steepest descent algorithm behaves much less evenly

Stopping criteria $\|u^{k^*} - u_0\|/\|u_0\| < eps$.

n, α	1.0e+00	1.0e-02	1.0e-04	1.0e-06	1.0e-08
8	2	3	60	108	8
16	2	2	33	190	155
32	2	2	31	279	253
64	2	3	33	398	374
128	2	3	39	538	513
256	4	5	44	673	665

But note that $\alpha \approx 10^{-6}$ is needed for $c_4^z/c_4^y \sim 1$.

Oneshot iterations on the boundary control problem

Setting initial guess to noise, using stopping criteria $\|r_{k^*}\| < 10^{-10}$.

n, α	1.00e+00	1.00e-02	1.00e-04
16	42	51	90
32	59	68	95
64	55	61	94
128	55	64	107
256	67	68	93

Steepest descent is less dependable

Setting initial control to noise, using stopping criteria $\|u^{k^*}\|/\|u^0\| < \epsilon$.

n, α	1.00e+00	1.00e-02	1.00e-04
16	126	164	3000
32	129	288	3000
64	121	1408	3000
128	76	1547	3000
256	35	909	336

Snippets from implementation (1/2)

```
# Define problem specific functionals:
def J(u,v): return (u-z)**2*ds + alpha*v**2*dx
def a(u,w): return u*w*dx + dot(grad(u),grad(w))*dx
def b(v,w): return (xi*v + (1-xi)*f)*w*dx + g*w*ds

# Build oneshot system by differentiation of Lagrangian:
L = J(u,v) + a(u,w) - b(v,w)
Lu = derivative(L, u, u_test)
Lv = derivative(L, v, v_test)
Lw = derivative(L, w, w_test)
Luw = derivative(Lu, w, w_trial)
# etc.

# Assemble each block of matrix separately:
Auw = assemble(Luw)
# etc.
```

Snippets from implementation (2/2)

```
1  # Combine matrices into full block matrix
2  A = block_mat([[Auu, Auv, Auw],
3                 [Avu, Avv, Avw],
4                 [Awu, Awv, Aww]])
5
6  # Define preconditioner norm, to be differentiated same way:
7  precnorm = (sqrt(alpha) * a(u,u)
8           +  sqrt(1.0/alpha) * a(w,w)
9           +  (u**2 + alpha*v**2 + (1.0/alpha)*w**2)*dx)
10 # ...
11 P = block_mat([[ML(Puu), 0, 0],
12                [0, ML(Pvv), 0],
13                [0, 0, ML(Pww)]])
14
15 # Solve block system with MinRes:
16 Ainv = MinRes(A, precond=P)
17 sol = Ainv*rhs
```

Questions?

- The FEniCS project: *http://www.fenicsproject.org*
- CBC.Block: *http://launchpad.net/cbc.block*
- *martinal@simula.no*